A note on Erdős covering systems

On distinct minimal covering systems with $2^n p$ as the LCM of the moduli

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In this article, we prove the existence of a distinct minimal covering system of congruences with $2^n p$ as the least common multiple of the moduli for all $n \geq p - 1$, where p is an odd prime, and we prove that there are $2^{p-1} p!$ distinct minimal covering systems of congruences with $2^{p-1} p$ as the least common multiple of the moduli.

Introduction

A collection of residue classes, $A = \{a_i \pmod{n_i} \mid 1 \leq i \leq k\}$ where $1 < n_1 \leq n_2 \leq \cdots \leq n_k$ is said to be a *covering system of congruences* [1] if every integer satisfies at least one of the congruences in A. And, A is called a *distinct covering system* [1] if $1 < n_1 < n_2 < \cdots < n_k$. Further, A is called a *minimal covering system* [2] if no proper subset of A is a covering system, and A is called a *disjoint covering system* [3] if every integer satisfies exactly one congruence in A. For convenience, we use the notation $a_i(n_i)$ to denote the congruence class $a_i \pmod{n_i}$ [3].

The concept of covering system of congruences was introduced by Erdős [4] in 1950. Erdős used the following distinct minimal covering system $\{0(2), 0(3), 1(4), 3(8), 7(12), 23(24)\}$ to create *an arithmetic progression of odd integers, no term of which is of the form $2^k + p$*, where p is a prime and k is a non-negative integer [4]. In 1975, Cohen and Selfridge [5] proved that *there exists an arithmetic progression of odd integers, no term of which can be expressed as the sum or difference of a power of two and a prime* using the following minimal covering systems $\{1(2), 0(4), 6(8), 10(12), 10(16), 18(24), 2(48)\}$ & $\{0(2), 0(3), 2(5), 5(9), 3(10), 4(15), 11(18), 1(20), 25(30), 17(36), 35(36), 31(60)\}$. The technique of devising a covering system of congruences has appeared histori-

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cally in this context (see [6, 7, 8]).

Selfridge [1] conjectured that *there is no distinct covering system with all moduli odd.*

Now, in 1950, Erdős conjectured that, *for any arbitrarily large c, there exists a distinct covering system whose minimum modulus is greater than c* [4]. As a consequence of this conjecture, distinct covering systems with a given minimum modulus were studied quite extensively. We refer the readers to Churchhouse (1968) [9], Krukenberg (1971) [10], Choi (1971) [11], Gibson (2009) [12], Nielsen (2009) [13], and Owens (2014) [14]. In 2015, Hough [15] disproved this conjecture of Erdős by proving that *the minimum modulus of a distinct covering system is at most* 10^{16}. In 2022, Balister et al. [17] lowered the bound of Hough [15] and proved that *the minimum modulus of a distinct covering system is less than* 616000.

Does there exist a distinct covering system with all moduli even? Yes, for example, {1(2), 2(4), 0(6), 4(8), 8(12), 16(24)} (see Nielsen [13]).

In 1971, Krukenberg [10] proved that for any odd prime p and for any natural number $n < p - 1$, *there is no distinct covering system with $2^n p$ as the least common multiple of the moduli.* He also proved that *there exists a distinct covering system with $2^{p-1}p$ as the least common multiple of the moduli.* In this article, we prove the existence of a distinct minimal covering system with $2^n p$ as the least common multiple of the moduli for all $n \geq p - 1$, and count the number of distinct minimal covering systems of congruences with $2^{p-1}p$ as the least common multiple of the moduli. Indeed, we have the following observations.

In 1967, Schinzel [16] conjectured that *every covering system of congruences has moduli n_i and n_j such that $n_i \mid n_j$ where $i \neq j$.* And, also proved that this conjecture is necessary for Selfridge conjecture.

Theorem 1. *Let p be an odd prime and $n \geq p - 1$ be any natural number. Then there exists a distinct minimal covering system with $2^n p$ as the least common multiple of the moduli.*

Lemma 1. *Let p be an odd prime and A be a distinct minimal covering system with $2^{p-1}p$ as the least common multiple of the moduli. Then the cardinality of A is $2p - 1$.*

In 2022, Balister et al. [17] proved the Schinzel conjecture. And hence, *there is no distinct covering system with all moduli prime.*

Theorem 2. *Let p be an odd prime. Then there are $2^{p-1}p!$ distinct minimal covering systems with $2^{p-1}p$ as the least common multiple of the moduli.*

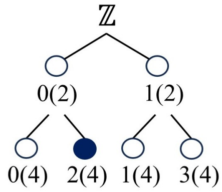
Corollary 1. *There are only 24 distinct minimal covering systems with cardinality 5.*

We recall the following lemmas, which will be useful while proving the above results.

Lemma 2 (see Corollary 2 of [18]). *Let A be a covering system and N be the least common multiple of the moduli of the congruences in A such that $N = \prod_{i=1}^{t} p_i^{\ell_i}$ where p_i's are distinct primes. If A is minimal, then* $|A| \geq \sum_{i=1}^{t} \ell_i(p_i - 1) + 1$.

Lemma 3 (see page 1 of [13]). *Let $A = \{a_i(n_i)\}_{i=1}^{k}$ be a distinct covering system. Then* $\sum_{i=1}^{k} \frac{1}{n_i} > 1$.

In 2009, Nielsen [13] used tree structure to visualize the congruence classes. For example, 2(4) is illustrated as follows.



1. Proof of Theorem 1

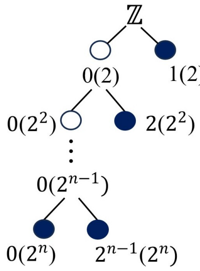
Let p be an odd prime and n be any natural number such that $n \geq p - 1$. We need to construct a distinct minimal covering system with $2^n p$ as the least common multiple of the moduli.

There is no distinct disjoint covering system (see [3]).

We first construct a distinct covering system, say $A = \{a_i(n_i)\}_{i=1}^{m}$, with $2^n p$ as the least common multiple of the moduli. Now since $n_i \mid 2^n p$, we have $n_i = 2^\ell, p,$ or $2^\ell p$, where $1 \leq \ell \leq n$.

Since $0(2^\ell) = 2^\ell(2^{\ell+1}) \cup 0(2^{\ell+1})$ and $2^\ell(2^{\ell+1}) \cap 0(2^{\ell+1}) = \emptyset$, it follows that $\{1(2), 2(2^2), \ldots, 2^{n-1}(2^n), 0(2^n)\}$ forms a disjoint minimal covering system (see Figure 1). Hence, the complement of $\cup_{j=1}^{n} 2^{j-1}(2^j)$ in $\mathbb{Z}$ is $0(2^n)$. Now the aim is to cover $0(2^n)$ using the leftover divisors of $2^n p$, namely, $p, 2p, \ldots, 2^n p$. If $x \equiv 0 \pmod{2^n}$, then there exists a unique integer $k \in \{0, 1, \ldots, p-1\}$ such that $x \equiv k2^n \pmod{2^n p}$. Thus, since $2^{n-k} p \mid 2^n p$, we have that $x \equiv k2^n \pmod{2^{n-k} p}$. Therefore, $\cup_{k=0}^{p-1} k2^n(2^{n-k} p)$ covers $0(2^n)$ and hence $A = \{2^{j-1}(2^j) \mid 1 \leq j \leq n\} \cup \{k2^n(2^{n-k} p) \mid 0 \leq k \leq p-1\}$ forms a distinct covering system of congruences with cardinality $n + p$ (illustrated in Figure 1).

A visualization of the disjoint cover $\{1(2), 2(2^2), \ldots, 2^{n-1}(2^n), 0(2^n)\}$



Now, we claim that A is minimal, or equivalently, for every $a_i(n_i)$ of A, there exists an integer x_i such that $x_i \equiv a_i \pmod{n_i}$ and $x_i \not\equiv a_m \pmod{n_m}$ for all $m \neq i$.

By letting $p = 3$ in the proof of Theorem 1, one can prove the existence of a distinct minimal covering system A with $|A| \geq 5$.

We first consider the case $a_j(n_j) = 2^{j-1}(2^j)$ for any $j = 1, 2, \ldots, n$. Then $2^{j-1} p \equiv 2^{j-1} \pmod{2^j}$, since $2^j \mid 2^{j-1}(p-1)$. Now, for any $m \neq j$, since $2^{j-1}(2^j) \cap 2^{m-1}(2^m) = \emptyset$, it follows that $2^{j-1} p \not\equiv 2^{m-1}$

(mod 2^m). Now, we claim that $2^{j-1}p \not\equiv k2^n \pmod{2^{n-k}p}$ for all $k = 0, 1, \ldots, p-1$. Suppose $2^{j-1}p \equiv k2^n \pmod{2^{n-k}p}$ for some k. Then $k2^n = 2^{j-1}p + r(2^{n-k}p)$ for some integer r, whence p divides k. This implies that $k = 0$ and thus, $2^{j-1}p \equiv 0 \pmod{2^n p}$, a contradiction.

There exist infinitely many distinct minimal covering systems with minimum modulus 2.

Now, consider the case $a_k(n_k) = k2^n(2^{n-k}p)$ for any $k = 0, 1, \ldots, p-1$. Clearly, $k2^n \not\equiv 2^{j-1} \pmod{2^j}$ for all $j = 1, 2, \ldots, n$. Now to complete the proof, we must show that for any $m \neq k$, $k2^n \not\equiv m2^n \pmod{2^{n-m}p}$. Suppose $k2^n \equiv m2^n \pmod{2^{n-m}p}$ for some $m \neq k$. Then, $p \mid (k-m)$ which implies that $k = m$, a contradiction. Thus, $A = \{2^{j-1}(2^j) \mid 1 \leq j \leq n\} \cup \{k2^n(2^{n-k}p) \mid 0 \leq k \leq p-1\}$ is a

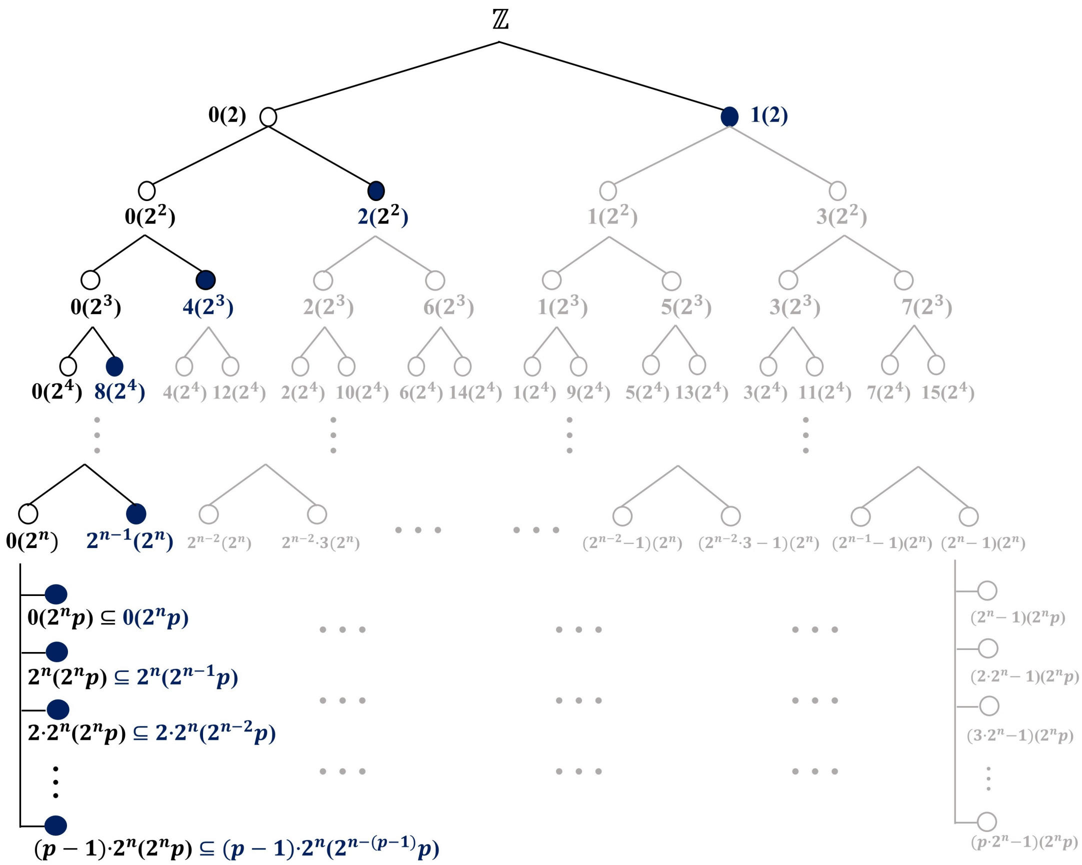


Figure 1. Existence of a distinct minimal covering system A with $2^n p$ as the LCM of the moduli.

distinct minimal covering system with $2^n p$ as the least common multiple of the moduli. This completes the proof. □

2. Proof of Lemma 1

Let A be a distinct minimal covering system with $2^{p-1}p$ as the least common multiple of the moduli. From the definition of the least common multiple, we see that any modulus n_i of A is a divisor of $2^{p-1}p$. Now, since $2^{p-1}p$ has $2p-1$ positive divisors that are greater than 1 and A being a distinct covering system, it follows that A can have at most $2p-1$ congruences. That is, $|A| \leq 2p-1$. In addition, since A is a minimal covering system, by Corollary 2 of [18], it follows that $|A| \geq 2p-1$. Thus, $|A| = 2p-1$. □

From Corollary 2 of [18], it follows that *there is no distinct minimal covering system with $p_1p_2p_3$ as the LCM of the moduli, where p_i's are distinct primes.*

3. Proof of Theorem 2

Let p be an odd prime and $\{a_i(n_i)\}_{i=1}^{k}$ be a distinct minimal covering system with $2^{p-1}p$ as the least common multiple of the moduli. Then by Lemma 1, $k = 2p-1$. Now since $n_i \mid 2^{p-1}p$, we have $n_i = 2^{\ell}, p$, or $2^{\ell}p$, where $1 \leq \ell \leq p-1$.

Let N be the least common multiple of the moduli of a distinct covering system, then $4 \mid \phi(N)$.

Now let $n_i = 2^i$ for all $i = 1, 2, \ldots, p-1$. Since there are two choices for each a_i (see Figure 2), there are clearly 2^{p-1} possibilities for the $(p-1)$-tuple $(a_1, a_2, \ldots, a_{p-1})$. And, we observe that there exists a unique integer $a \in \{0, 1, \ldots, 2^{p-1}-1\}$ such that the complement of $\cup_{i=1}^{p-1} a_i(2^i)$ is $a(2^{p-1})$ (see Figure 2).

Now let $n_{(p-1)+i} = 2^{i-1}p$ for all $i = 1, 2, \ldots, p$. Now we claim that there are $p!$ ways to cover $a(2^{p-1})$. To see this, if $x \equiv a \pmod{2^{p-1}}$, then there exists a unique integer $j \in \{0, 1, \ldots, p-1\}$ such that $x \equiv a + j2^{p-1} \pmod{2^{p-1}p}$. Now since $n_i \mid 2^{p-1}p$, we have $x \equiv a + j2^{p-1} \pmod{n_i}$. Thus to cover $a(2^{p-1})$, it is enough to cover the following p integers, namely, $a + j2^{p-1}$ for all $j = 0, 1, \ldots p-1$. Thus, there exist p choices for a_p, $p-1$ choices for a_{p+1}, and so on, and hence there are $p!$ possibilities for the p-tuple $(a_p, a_{p+1}, \ldots, a_{2p-1})$.

Suppose there exists $j_1, j_2 \in \{0, 1, \ldots, p-1\}$, $j_1 \neq j_2$ such that $a + j_12^{p-1} \equiv a + j_22^{p-1} \pmod{2^kp}$
$\implies p \mid (j_1 - j_2)2^{p-1}$
$\implies p \mid j_1 - j_2$
$\implies j_1 = j_2$
$\Rightarrow\Leftarrow$

Therefore, we have shown that there are $2^{p-1}p!$ distinct minimal

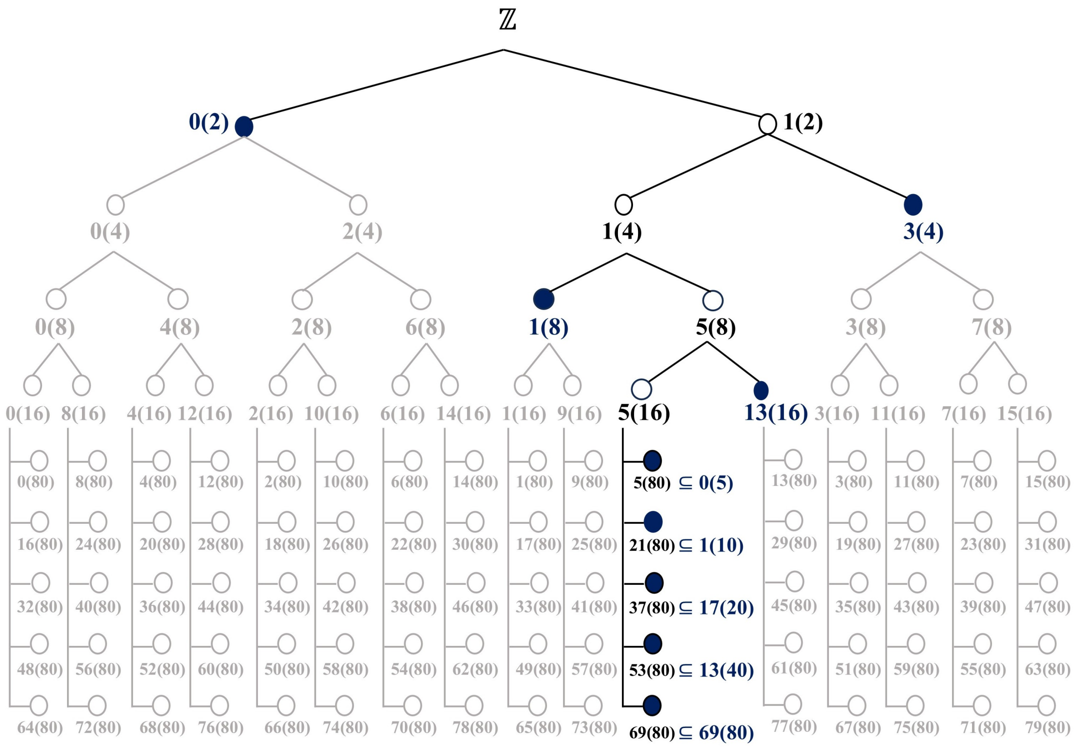


Figure 2. A distinct minimal covering system with 80 ($= 2^4 \cdot 5$) as the LCM of the moduli.

covering systems with $2^{p-1}p$ as the LCM of the moduli. □

4. Proof of Corollary 1

Let $A = \{a_i(n_i)\}_{i=1}^{k}$ be a covering system. If A is disjoint, then $\sum_{i=1}^{k} \frac{1}{n_i} = 1$ (see [3]). If A is distinct, then $\sum_{i=1}^{k} \frac{1}{n_i} > 1$ (see [13]).

Let $A = \{a_i(n_i)\}_{i=1}^{5}$ be a distinct minimal covering system with cardinality 5, and N be the least common multiple of the moduli. Let $N = \prod_{i=1}^{r} p_i^{\ell_i}$ where $2 \leq p_1 < p_2 < \cdots < p_r$ are primes and ℓ_i's are positive integers. Since $\sum_{i=1}^{5} \frac{1}{n_i} > 1$ (see page 1 of [13]), it follows that there is no distinct covering system with $N = p^{\ell}$, and thus $r \geq 2$. Now we claim that $r = 2$. For, if $r \geq 3$, then by Corollary 2 of [18], it follows that

$$
\begin{aligned}
|A| &\geq \sum_{i=1}^{r} \ell_i \, (p_i - 1) + 1 \\
&\geq (p_1 - 1) + (p_2 - 1) + (p_3 - 1) + 1 \\
&\geq 1 + 2 + 4 + 1 = 8 \qquad (\text{since } 2 \leq p_1 < p_2 < p_3)
\end{aligned}
$$

which is a contradiction to $|A| = 5$. Hence, $r = 2$ and $N = p_1^{\ell_1} p_2^{\ell_2}$. Next, we claim that $p_1 = 2$ and $p_2 = 3$. For, if $p_1 \geq 3$, we have $p_2 \geq 5$. Then by Corollary 2 of [18], $|A| \geq (p_1 - 1) + (p_2 - 1) + 1 \geq 7$, a contradiction. A similar argument applies when $p_1 = 2$ and $p_2 \geq 5$. Thus, $p_1 = 2$ and $p_2 = 3$. Therefore, $N = 2^{\ell_1} 3^{\ell_2}$.

Now we claim that $\ell_1 = 2$ and $\ell_2 = 1$. For if $\ell_1 = \ell_2 = 1$, then $N = 6$, whence $|A| \leq 3$ since 6 has only three positive divisors that are greater than 1, a contradiction. For if either $\ell_1 \geq 3$ or $\ell_2 \geq 2$, then it follows from Corollary 2 of [18] that $|A| \geq 6$, a contradiction. Thus, $N = 2^2 \cdot 3 = 12$. By Theorem 2, there are $2^2 \cdot 3!$ (= 24) distinct minimal covering systems with 12 as the least common multiple of the moduli. □

The list of distinct minimal covering systems with cardinality 5:

{0(2), 0(3), 1(4), 1(6), 11(12)}
{0(2), 1(3), 3(4), 3(6), 5(12)}
{1(2), 0(3), 0(4), 2(6), 10(12)}
{1(2), 1(3), 2(4), 0(6), 8(12)}
{0(2), 0(3), 1(4), 5(6), 7(12)}
{0(2), 1(3), 3(4), 5(6), 9(12)}
{1(2), 0(3), 0(4), 4(6), 2(12)}
{1(2), 1(3), 2(4), 2(6), 0(12)}
{0(2), 0(3), 3(4), 1(6), 5(12)}
{0(2), 2(3), 1(4), 1(6), 3(12)}
{1(2), 0(3), 2(4), 2(6), 4(12)}
{1(2), 2(3), 0(4), 0(6), 10(12)}
{0(2), 0(3), 3(4), 5(6), 1(12)}
{0(2), 2(3), 1(4), 3(6), 7(12)}
{1(2), 0(3), 2(4), 4(6), 8(12)}
{1(2), 2(3), 0(4), 4(6), 6(12)}
{0(2), 1(3), 1(4), 3(6), 11(12)}
{0(2), 2(3), 3(4), 1(6), 9(12)}
{1(2), 1(3), 0(4), 0(6), 2(12)}
{1(2), 2(3), 2(4), 0(6), 4(12)}
{0(2), 1(3), 1(4), 5(6), 3(12)}
{0(2), 2(3), 3(4), 3(6), 1(12)}
{1(2), 1(3), 0(4), 2(6), 6(12)}
{1(2), 2(3), 2(4), 4(6), 0(12)}

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